Derivation of the Weibull distribution based on physical principles and its connection to the Rosin–Rammler and lognormal distributions

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We describe a physically based derivation of the Weibull distribution with respect to fragmentation processes. In this approach we consider the result of a single-event fragmentation leading to a branching tree of cracks that show geometric scale invariance (fractal behavior). With this approach, because the Rosin–Rammler type distribution is just the integral form of the Weibull distribution, it, too, has a physical basis. In further consideration of mass distributions developed by fragmentation processes, we show that one particular mass distribution closely resembles the empirical lognormal distribution. This result suggests that the successful use of the lognormal distribution to describe fragmentation distributions may have been simply fortuitous.

I. HISTORICAL BACKGROUND

In 1933 Rosin and Rammler[1,2] proposed the use of an empirical distribution for description of particle sizes, which they obtained from data describing the crushing of coal and other materials. In 1939 Weibull[3] proposed the same distribution (as we show below), which he obtained from the study of the fracture of materials under repetitive stress. The distribution proposed was strictly empirical,[4] until Austin *et al.*[5] derived it to describe batch grinding in 1972. Later, Peterson *et al.*[6] and Brown,[7] and Wohletz *et al.*[8] independently rederived the distribution. Austin *et al.*, Peterson *et al.*, and Brown each derived the distribution from a somewhat different point of view, but they all used a simple but nonetheless *empirical* power law to describe the breakup of a single particle into smaller particles. In this article we eliminate this shortcoming and thus put the Weibull distribution on a solid theoretical basis, stemming from physical principles.

II. DERIVATION OF THE WEIBULL DISTRIBUTION

Brown[7] began his theory of sequential fragmentation with the equation

$$n(m)=C\int_m^\infty n(m')f(m'\rightarrow m)dm'. \tag{1}$$

Here $n(m)$ is the number distribution in units of particles per unit mass of mass m between m and $m+dm$. $f(m'\rightarrow m)$ is the single-event particle distribution function and expresses the distribution in mass, m, arising from the fragmentation of a single, more massive particle of mass m'. Equation (1) represents the summing of all contributions to the distribution at m from the fragmentation of all particles of mass $m'>m$.

Brown[7] set the constant C equal to m_1^{-1}, and chose

$$f(m'\rightarrow m)=\left(\frac{m}{m_1}\right)^\gamma, \tag{2}$$

where $-1<\gamma\leq 0$.

Inserting Eq. (2) into Eq. (1), we have

$$n(m)=\left(\frac{m}{m_1}\right)^\gamma\int_m^\infty n(m')d\left(\frac{m'}{m_1}\right). \tag{3}$$

The solution to Eq. (3) is

$$n(m)=\frac{N_T}{m_1}\left(\frac{m}{m_1}\right)^\gamma \exp\left[-\frac{(m/m_1)^{\gamma+1}}{\gamma+1}\right], \tag{4}$$

which is the Weibull distribution in particle number. Equation (4) has been normalized such that

$$N_T=\int_0^\infty n(m)dm, \tag{5}$$

where N_T is the total number of fragments in the distribution.

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3/11/2007

Dear Irene: I accept the challenge of trying to explain my fragmentation theory:
First consider m'--->m where m' is the larger mass and m is the result after an unspecified process. So, I call that (m'-->m) the fragmentation of a single particle. I'll call the unknown process f(m'-->m).

Now we need to know how many particles started at the "top" of this process, so we can see how many smaller (processed) particles come out the "bottom". I call this unknown "feed" function, n(m'), the number of particles of large mass m' starting out at the top. So, so far we have n(m')f(m'-->m).

And now we need to know the sum result of many different m' particles of various large masses undergoing this process. So we must do a summation over all m' particles! (in math this is called an "intergration", and is symbolized as a large ess-like symbol. I'll have to write it in by hand later.) We must integrate from the smallest particles, m, up to the maximum m' paticles.For simplicity I just set this upper "integral limit" at infinity (that I will also have to write in later) (We know that no particles can actually be infinitly massive, but I do that bearing in mind that it is unrealistic). We must integrate (include) all the m' particles, so we are integrating over all m' paticles from mass m to infinity.

The equation looks like this so far:

$$\int_m^{\infty} n(m')f(m' \rightarrow m)\,dm'$$ (Where the dm' means the sum is done over all m')

This gives the total number of m' particles processed down to mass m (we are near the end)

So we can set $$n(m) = C \times \int_m^{\infty} n(m')f(m' \rightarrow m)\,dm'$$

Where C is just a constant denoting our lack of knowledge of the true total (we'll fix that)

And that's ALL! notice that we have not specified the exact process (so it's "free")

Now look at equation (1) in the article, and Lo! you recognize it!

That's why I was "stunned" when I looked at the simplicity of it, but unspecificity!
It can therefore be applied to ANY kind of fragmentation, and that gives it a surprising generality, and therefore a wonderful versatility! Fantastic!

However, note that that the function n appears both inside the integral and outside, this makes it an "integral equation" that are notoriously difficult to solve. And so is this one! (we finally asked a mathematician friend of mine, help us sort out our several attempts.)

Love, Bill

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From: <wkbrown@inreach.com>
To: "Jane Braxton Little" <jblittle@dyerpress.com>
Cc: "Bill Brown" <wkbrown@inreach.com>
Sent: Wednesday, October 17, 2007 5:09 PM
Subject: How I started this divertissement

Dear Jane:
At the risk of boring you to tears, I will tell you how this adventure started. Part of it is in my Autobiography that you
have, part is in my first, long article that I'll send you now. It contains the reference to the Document that you sent me on "Operation RollerCoaster" (You must either have a fantastic memory, filing system, and be truly organized! (What did I experct of a collaborator of Sally Posner?) OK, you have my original article (too long to read, I fear.)

It began during my final couple of years at Los Alamos Lab. I had had my present job for some 5-6 years, and after
that much time in the same job I was (as before in other Lab Jobs) bored. So I went spooking around to see what others might be doing. I soon discovered Ken Wohletz in his basement lab in the Physics Building. Ken is a Vulcan-
ologist, and was measuring the mass distribution of his many personally-gathered volcanic ash samples. This did get my attention! I didn't even know what a mass distribution WAS, and certainly not why anyone would measure them. He taught me quickly. His setup consisted of a stack of sieves, coarse ones of the top, fine ones on the bottom. He would put his sample in the top, shake the stack, and the remove the trays one by one, measuring the increase in mass of each and thus thus obtaining the
distribution of masses in the ash sample. Simple! Even for me, a neophyte! His data was then in the form of a bar graph, and he fitted that bar graph with an empirical curve of one type or another. (This later bothered me, because
an empirical curve (empirical=from the data, and holds no intrisic meaning). But to continue, I began sneaking off to his basement lab whenever I had the chance, to participate in the fun.
Now: After the War, most of the scientists and engineers who built The Bomb (and ended the war), return ASAP to
their Universities and Laboratories, to continue their own research. Although Los Alamos itself, in a pine forest, is a
beautiful place to work (Oppenheimer knew what he was doing), there was difficulty in attracting new scientists and
engineers to this isolated hilltop (that most people envisioned as a desert, it being in New Mexico). So the Management, I.e., Oppenheimer, instituted some very attractive extras, like high pay, and month-long vacations! I was working in a Danish lab at the time, but I had been there for two years, and felt that, although Kitty and I loved
Denmark and the danish people (and were soon adopted as fellow Danes),it was time to start job-shopping in the U.S. I had heard glowing reports of Los Alamos and seen pictures of Los Alamos, and I applied there first. The Lab sent a Staff Member (John Hopkins) over from England to Interview my at the Risoe Laboratory where I was a
member of a Danish Team, measuring the free neutron half-life. I was amazed and flattered! However, John and I were instantly simpatico, and I got the Los Alamos Job! Heaven! My family and I immediately began to save every possible vacation day to spend camping at Lake Almanor, on a lot we had previously bought. More Heaven! We set
up our tents at water's edge and loved the sound of lapping waves. We would build a cooking pit of local rock, and
cook our dinners by the lake every evening. Soon I got the idea of reading to our two boys by the stoked up fire after
dinner. I started off easy with L.Frank Baum's OZ Books, and the boys loved it. I moved on to other childrens books:
Tarzan, etc. One year, consuming reading material at a prodidgious rate, I read them J.R.R.Tokien's "The Hobbit".
This really captivated our boys! In the succeeding year, I considered tackling "The Lord of the Rings Trilogy", but the
geography, politics, and Evil daunted me. The boys clamored for more Tolkien, so I bought a map of Middle Earth

and we dove into this complexity. I knew that both boys were bright, and I judged that they could handle it, and did they ever! They would run to their tent (where they and our wonderful dog, "Moose" lived), and come out yelling "Start
here, Dad, this is where we left off. No one could resist such adulation, and I was no exception. I think I enjoyed The
Lord of The Rings as much as they did! We finished the Trilogy that Summer, and nothing would do, but to repeat it
the following year! What a Time we had during those years! A few years ago, I presented Walt, the afficianado, with
the four tattered books,and he has carefully kept them all these years! It brought our little family quite close together,
and a few years ago, from the porch of our house there, I looked down at that place, and tears came to my eyes...
But I started off to relate how Sequential Fragmentation started: One Summer, in our half-built house, while Kitty, the
two boys, and Moose were down at the beach, I sat at the kitchen counter after lunch and continued worrying about
the use of using the several empirical curves for data-fitting. Surely, I thought, there must be some meaning behind those curves that worked so well! So I attempted to see if I could possibly write down something in mathematcal
terms to explain their manifest success... I doodled a thought: (m'-->m) where m' is a large mass and m is a smaller mass.That much was obvious. But that started me thinking harder. I said to myself: If I want to put the m' s
though a process,I would have to know how many large particles that I fed in at the start. (I can only guess what I must have been thinking, because too many years have passed and my memory is poor, at best). So I decided to name the number of large particles fed in: "n(m')", thus I had n(m')f(m'-->m), & insight came flashing into my mind! Now, I must sum over all the particles of m' to find out how many wind up at m ! I quickly finished the thought by completing Equation(1), then just sat there, stunned, and stared at what I had written. It was simple, yet deep! I must have sat there staring at that enigmatic equation for half an hour! I saw no immediate connection with the
empirical curves that had worried me, but I knew instinctively that I had something important in front of me, and that
it was profound! Indeed I did, as subsequent developments revealed.I knew that I needed a suitable function to use
for f(m'--m) and that it must be simple, but also have the property that it express the fact that I would have more m particles at the end of the process, and start with less m' particles at the beginning.The obvious simple function with these properties,I intuitively felt, must be a power funtion, equation(2).Later, this choice gave rise to a great deal
of agonizing, when the more theoretical members of our team stated that I could have used no other function, and that the geometry of the situation required a power function. I was, frankly surprised and somewhat dazed when the proof was presented! Accepting that, albeit tentatively, I ran headlong into a nasty "Integral equation"(3) that had the
function n both inside and outside the integral sign (that expressed the summing that I mentioned earlier.) I tried hard, but was totally defeated by this integral equation (3).In desperation I managed to divide it up into a few smaller
integrals that I hoped to integrate, and then somehow collect into a solution. At a loss, I wrote this up in a finished paper, not mentioning my dilemma. I submitted it to the editor of the journal (who is not concerned with validity, but
merely passed my paper to two competent mathematicians). The same terse answer came from both of them: "Brown's aproach is hopeless, but the solution is......"I looked at both the offered solutions and rejected one that I could plainly see was wrong, but the other looked promising. I plugged that solution into the integral equation, and it satisfied it -- it was indeed the solution! So I wrote it into the paper and resubmitted it. It was accepted for publication! I am both a physicist and an engineer, and while the engineer "side" of me was satisfied, the physicist "side" was queasy. Then, looking at the solution, I recognized it as The Weibull Dstribution, that had been used with
great empirical success for many years. Here was my connection that I had hoped for Equation(1)! Delirium! Our team,: Ken Wohletz, myself & Akihiro Hori have spent much labor proving that what I had published was true without
any doubt. I even soliicited the help of a true mathematician, John Brillhart, with whom I had sung years before in the Men's Glee Club at Berkeley, and who had spent his career as a number theorist at the University of Arizona in
Tucson. He solved the integral equation (3), starting from scratch, with alacrity, and found the single, simple error

that Ken had made in the attempt he had made in his attempt to solve the integral equation! Afterwards John told me that "He would have nothing more to do with physical science", and returned to his first love, of number theory.

So ends my story of how I supplied a Physical Basis for The Weibull Distrution, and formulated The Sequential Fragmentation Theory". I'll probably never again experience such joy, excitement, and satifaction in my lifetime....

SEQUENTIAL FRAGMENTATION THEORY

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A couple of months ago I received a surprise email from A Japanese Mining Engineer, named Akihiro Hori. DR. Hori said that he had noticed my Sequential Fragmentation Theory (SFT), and that the particle distibution produced by my theory gave a near-perfect fit to the distribution he obtains after the grinding and crushing of his ores! I had published my theory 15-20 years ago, and since had lain idle so long that I assumed that it was dead, and forgot about it! "Aki" (as he wishes to be called) asked for help in understanding my theory, and, of course, I assented. This took some re-learning on my part, particularly on the initial, rather enigmatic equation. I time I was able to provide a lucid explanation of that equation, and wrote a "simplified" derivation of it.

Aki explained that he considered the knowledge of the use of my theory so important that his goal was to spread the knowledge throughout Japan! I promised to help, and we began working together. Aki plans to complete a Web Page where all such knowledge can be placed for ready access to anyone (in English, later in Japanese). We have now verified each step in reaching in reaching the solution to my theory, which is, in fact, the Weibull Distribution. This distribution has long enjoyed successful empirical use, but is now on a solid physical basis. Aki and I have become close personal friends, and we have spoken to together over the telephone. At present we are still writing and collecting information to be placed on Aki's Web Page. Our third colleague, Dr. Kenneth Wohletz, a vulcanologist, is at present enmeshed in Los Alamos Laboratory affairs, but will soon rejoin us.

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